

### 6.3 One-Sample Mean (Sigma Unknown)

In the previous section we examined the hypothesis of comparing a sample mean with a population mean with both the population mean and standard deviation well known. In practice, most hypothesis involving population parameters in which the population parameters are known are rare. Often we must use sample statistics to estimate and test hypothesis statement about population parameter. When the population standard deviation is known we observed that the Central Limit Theorem that the sampling distribution's standard deviation is the same as the standard error of the sampling mean,  $s_M$ , where

$$\sigma_M = \frac{\sigma}{\sqrt{n}}, \text{ where } \sigma \text{ is the population standard deviation}$$

But when the population variance is not known, the sample variance (standard deviation squared) may be used to estimate the population variance. Since the sample variance has  $n - 1$  degree of freedom, it will be larger than the true population variance, but will be close to the true population variance when the sample size gets very large. Recall that the sample standard deviation is the square root of the variance and is given by the formula below:

$$\text{sample standard deviation} = s = \sqrt{\text{sample variance}}$$

$$s = \sqrt{s^2} = \sqrt{\frac{SS}{n-1}} = \sqrt{\frac{SS}{df}}, \text{ where } SS \text{ is the sum of square deviation}$$

The sampling distribution's standard error when the sample standard deviation is used to estimate it is therefore

$$\text{estimated standard error} = s_M = \frac{s}{\sqrt{n}}, \text{ where } s \text{ is the sample standard deviation}$$

The **estimated standard error** ( $s_M$ ) is uses the sample standard deviation with degree of freedom,  $n - 1$  to estimate the true sampling standard error ( $s_M$ ) when the population standard deviation,  $s$  is not known.

### The $t$ Distribution

The standard normal distribution is used as the underlying sampling distribution when the population standard deviation is known; however, when it is unknown, we use the  $t$  distribution introduced here.

The  $t$ -distribution is similar to a normal distribution and approaches that of a normal curve when the sample size is very large. The **degree of freedom** ( $df$ ) is the number of independent scores of a sample remaining after estimating one or more population parameters. The degree of freedom is related to the sample size.

**Degree of freedom** ( $df$ ) is the number of scores that are independent and allow to vary after estimating a population parameter. It is the number of observations minus the number of restrictions or observations that are restricted.

The sampling distribution of the  $t$  statistics is called the **Student's  $t$  distribution**.

The following are properties of the  $t$ -distribution:

1. *There are an infinite number of student  $t$  distributions*, one for each degree of freedom,  $df$ . The number of **degrees of freedom** for the One-sample case is 1 minus the sample size or  $n - 1$ . That is,  **$df = n - 1$** .

2. *The Student's  $t$  distribution is similar to the standard normal distribution in shape*: That is it as a mean of  $t = 0$  and the total area under the curve (or the probability density or distribution of the function) is 1.

3. *As the sample size gets large the curve gets closer to the standard normal curve*. Therefore for sample size = 30 the  $z$ -score statistics can be used.

There are many  $t$  distributions for each degree of freedom. Figure 6.3.1 shows a comparison between a  $t$  distribution and the normal distribution.

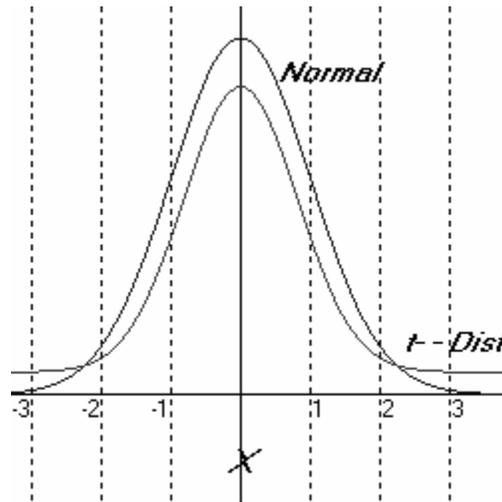


Figure 6.2.1 Normal and t-distribution curves

### Hypothesis Testing: One-Sample Case for the Mean

**One-Sample Case:** If you wish to compare the mean of a sample to a test value, a constant, you use the  $t$  distribution. For this test you will need the sample mean,  $M$ ; the sample standard deviation,  $s$ , the sample size,  $n$  and the value that you want the sample mean compared to. The sample mean is often compared against a hypothesized or known population mean,  $\mu$ .

**Hypotheses:** Your *null hypothesis*,  $H_0$  (test statement) is that, your sample mean equals some test value. Your alternate hypothesis,  $H_a$  is that there is no reason to believe that the sample mean differs from the test value,  $E$  or a hypothesized population mean,  $\mu$ .

$H_0: \mu = E$  or  $\mu - E = 0$ , where  $E$  is the test value

As before, the alternative hypothesis can be stated in one of three ways:

$H_a: \mu \neq E$  (non-directional) or

$H_a: \mu > E$  (directional) or

$H_a: \mu < E$  (directional).

**The  $t$  Statistics:** The  $t$  statistics is similar to the  $z$  statistics, except that the estimated standard error,  $s_M$  is used to estimate  $s_M$ . The  $t$  statistics is simply

$$t \text{ statistics} = t = \frac{M - \mu}{s_M} = \frac{M - \mu}{\frac{s}{\sqrt{n}}}$$

$M$  is the sample mean and  $\mu$  is the hypothesized mean or test value.

Just as we were able to use the normal distribution table to find proportions or probabilities associated with the  $z$  statistics, we will use the  $t$  distribution table in the appendix (Tables A2 & A3) to find proportions or probabilities for the  $t$  statistics. Table A2 is a two-tailed  $t$  probability distribution table at alpha levels of 0.05 (5%) and 0.01 (1%) at various degree of freedom. Table A3 is a one-tailed  $t$  probability distribution table at various alpha levels and degree of freedom. When the degree of freedom,  $df$ , is large, the normal probability distribution table can be used. Let us now examine the hypothesis testing procedure involve in make inference about a hypothesized population mean from sample estimates. These three problems explore the procedure for the various alternative hypothesis statements.

**Problem statement 1:** The output of a manufacturing process is monitored by taking a sample size of  $n = 25$  vials to determine the level of an important ingredient. If the amount of the ingredient is not equal to exactly 0.05 gram per liter, the process will be stopped and the tanks purged. Otherwise, the process will continue. The mean from the sample is 0.064 with a standard deviation of 0.017. At a significant level of  $\alpha = 0.01$ , should the process be stopped and the tanks purged?

**Step 1: State the hypotheses**

$$H_0: \mu = 0.05 \text{ and } H_a: \mu \neq 0.05 \text{ (non-directional)}$$

**Step 2: Select the significance level**

$$\alpha = 0.01 \text{ (1\%, two-tailed with 0.5\% in each tail)}$$

**Step 3: Compute the test statistics**

$$\text{standard error} = s_M = \frac{s}{\sqrt{n}} = \frac{0.017}{\sqrt{25}} = 0.0034$$

$$t \text{ statistics} = t = \frac{M - \mu}{s_M} = \frac{0.064 - 0.05}{0.0034} = 4.1176$$

**Step 4: Determine the critical region**

The critical region for the  $t$  distribution with  $df = 24 = (n - 1)$  and  $\alpha = 0.01$  (two-tailed, Table A2) is  $t = -2.797$  or  $t = 2.797$  (i.e., reject  $H_0$  if  $t$  statistics is within these regions)

**Step 5: Make a decision**

We **reject** the null hypothesis for any of the following reasons:

1. The test statistics  $t = 4.1176$  is within the critical region:  $t = 4.12 > t_{CV} = 2.797$  **or**
2. The  $p$ -value for the test is **0.004** (from any statistics program)  $< 0.01$  **or**
3. The hypothesized  $\mu = 0.05$  is outside the sample  $CI_{95}$  of **0.055** to **0.074**  
 $CI_{95} = 0.064 \pm 2.797(0.0034)$  **or**
4. The  $CI_{95}$  for the difference does not contain zero:  $H_0: \mu - E = 0$   
 $CI_{95} = 0.014 \pm 2.797(0.0034) = 0.0045$  to **0.0235**  
 $Mean \text{ Difference} = M_D = 0.064 - 0.05 = 0.014$

The effect size is  $d = 0.82$  (**large effect**,  $d = 0.8$ )

$$\text{effect size} = d = \frac{M - \mu}{s} = \frac{0.064 - 0.05}{0.017} = 0.8235$$

So the manufacturing process has changed and so the tanks should be purged. There is significant difference between the sample mean and the hypothesized population mean.

**Problem statement 2:** Test the hypothesis, at a significance level of  $\alpha = 0.05$ , that the average enrollment for the math department (across all its classes) has **increased** from an average enrollment of 160. A recent sample of 30 math classes show the average enrollment is 170 with a standard deviation of 30.

**Step 1: State the hypotheses**

$$H_0: \mu = 160 \text{ and } H_a: \mu > 160 \text{ (directional)}$$

**Step 2: Select the significance level**

$$\alpha = 0.05 \text{ (5\%, upper one-tailed)}$$

**Step 3: Compute the test statistics**

$$\text{standard error} = s_M = \frac{s}{\sqrt{n}} = \frac{30}{\sqrt{25}} = 6.0$$

$$t \text{ statistics} = t = \frac{M - \mu}{s_M} = \frac{170 - 160}{6} = 1.6667$$

**Step 4: Determine the critical region**

The critical region for the  $t$  distribution with  $df = 24 = (n - 1)$  and  $\alpha = 0.05$  (one-tailed, Table A3) is  $t = 1.711$  (i.e., reject  $H_0$  if  $t$  statistics is  $\geq 1.711$ ,  $t$  critical value)

**Step 5: Make a decision**

We Do not reject the null hypothesis for any of the following reasons:

1. The test statistics  $t = 1.67$  is not in the critical region ( $t = 1.67 < t_{CV} = 1.71$ ) **or**
2. The  $p$ -value for the test is **0.054** (from any statistics program)  $> 0.05$  **or**
3. The hypothesized  $\mu = 160$  is within the sample  $CI_{95}$  of **157.62** to **182.38**  
 $CI_{95} = 170 \pm 2.064(6)$  [ $t_{CV} = 2.064$ :  $df = 24$ ,  $\alpha = 0.05$ , two-tailed] \*
4. The  $CI_{95}$  for the difference does contain zero:  $H_0: \mu - E = 0$   
 $CI_{95} = 10 \pm 2.064(6) = -2.38$  to **22.38**  
 $Mean \text{ Difference} = M_D = 170 - 160 = 10$

The effect size is  $d = 0.33$  (**medium effect**,  $0.2 < d < 0.6$ )

$$\text{effect size} = d = \frac{M - \mu}{s} = \frac{170 - 160}{30} = 0.3333$$

The math department's enrollment average has not changed significantly from 160.

\* Note: the two-tailed critical value is used to compute the confidence interval

**Problem statement 3:** A sample of 8 observations averages 5.8 and the standard deviation is 2.034. Is this mean **less than** the existing mean of 7.0? Test this hypothesis using a 5% level of significance.

**Step 1: State the hypotheses**

$$H_0: \mu = 7.0$$

$$H_a: \mu < 7.0 \text{ (directional)}$$

**Step 2: Select the significance level**

$$\alpha = 0.05 \text{ (5\%, two-tailed with 2.5\% in each tail)}$$

**Step 3: Compute the test statistics**

$$\text{standard error} = s_M = \frac{s}{\sqrt{n}} = \frac{2.034}{\sqrt{8}} = 0.7191$$

$$t \text{ statistics} = t = \frac{M - \mu}{s_M} = \frac{5.8 - 7.0}{0.7191} = 1.6687$$

**Step 4: Determine the critical region**

The critical region for the  $t$  distribution with  $df = 7 = (n - 1)$  and  $\alpha = 0.05$  (one-tailed, Table A3) is  $t = 1.895$  (i.e., reject  $H_0$  if  $t$  statistics is  $> 1.895$ ,  $t$  critical value)

**Step 5: Make a decision**

We Do not reject the null hypothesis for any of the following reasons:

5. The test statistics  $t = 1.6687$  is not in the critical region ( $t = 1.67 < t_{CV} = 1.895$ ) **or**
6. The  $p$ -value for the test is **0.07** (from any statistics program)  $> 0.05$  **or**
7. The hypothesized  $\mu = 7.0$  is within the sample  $CI_{95}$  of **4.1 to 7.5**  
 $CI_{95} = 5.8 \pm 2.3646(0.7191)$  [ $t_{CV} = 2.3646$ :  $df = 7$ ,  $\alpha = 0.05$ , two-tailed]
8. The  $CI_{95}$  for the difference does contain zero:  $H_0: \mu - E = 0$   
 $CI_{95} = -1.2 \pm 2.3646(0.7191) = -2.9$  to  $0.50$   
 $Mean \text{ Difference} = M_D = 5.8 - 7.0 = -1.2$

The effect size is  $d = 0.59$  (**medium effect**,  $0.2 < d < 0.6$ ; use absolute value of result)

$$\text{effect size} = d = \frac{M - \mu}{s} = \frac{5.8 - 7.0}{2.034} = -0.59 = +0.59$$

The average is not significantly less than 7.0.

### Acceptance and Rejection Criteria:

**Reject the null hypothesis** for any of the following reasons:

1. The significance level  $<$  alpha or  $p$ -value  $<$  0.05
2. The absolute value of the  $t$ -test (calculated)  $>$   $t$ -value (table lookup)
3. The confidence interval for the mean difference does *not* contain zero.

Do not reject (**accept**) the null hypothesis for any of the following reasons:

1. The significance level  $>$  alpha or  $p$ -value  $>$  0.05
2. The absolute value of the  $t$ -test (calculated)  $<$   $t$ -value (table lookup)
3. The confidence interval for the mean difference contains zero

The SPSS output for the One-sample  $t$ -test will provide sufficient information for you to make a decision about your null hypothesis. The following are two examples of the one-sample  $t$ -test. The SPSS procedure to calculate the one-sample  $t$ -test for a single variable and compare it to a constant value is as follows: **Analyze -> Compare Means -> One-Sample  $t$  Test** -> Enter Test Value and under Option Select Confidence Interval of 95% (for alpha of 0.05 or significance level). Figure 6.2.2 shows the SPSS procedure for computing the one-sample  $t$ -test for mean using sample data against a hypothesized population mean.

**Example 1:** Is the mean of the *pass9th* variable 70? Since we wish to compare the mean of a single variable to a constant value of 70, we can use the one-sample  $t$ -test with an alpha or  $\alpha$  of 0.05 (5%). When we performed this analysis using SPSS we get the basic statistics summary in Table 6.2.1.

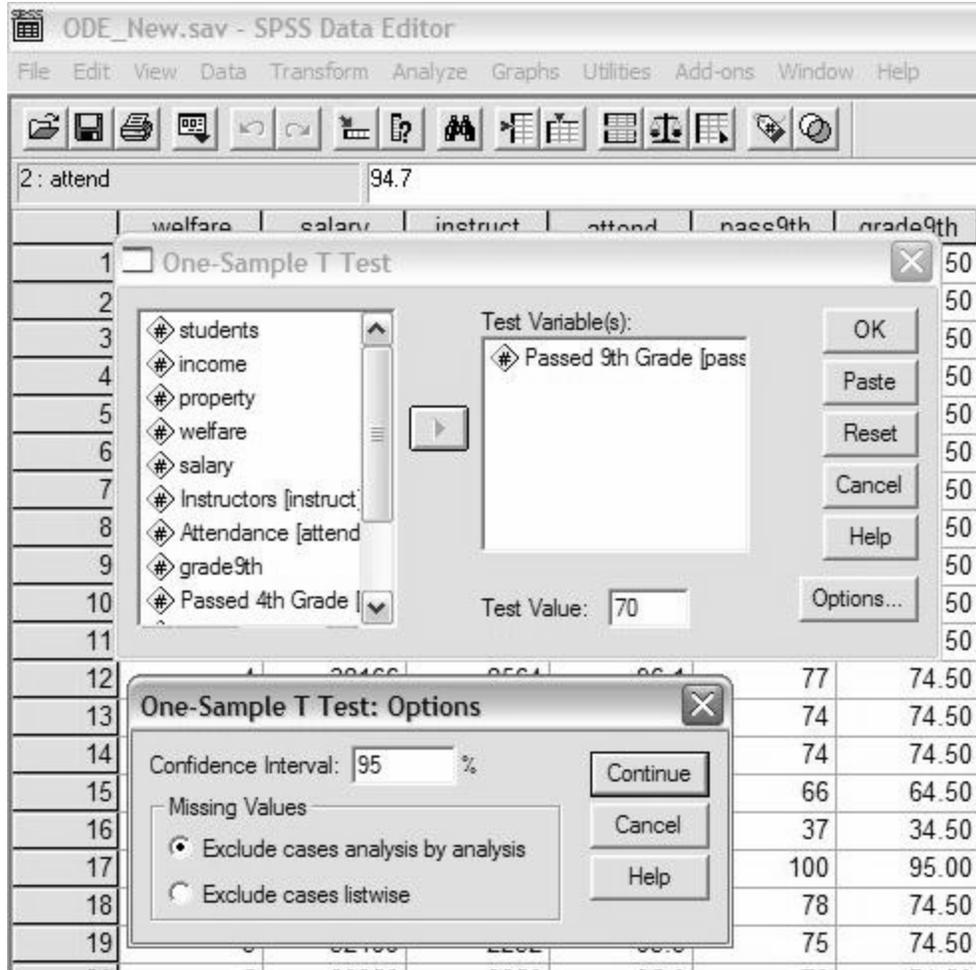


Figure 6.2.2. SPSS one-sample t-test procedure

Table 6.2.1 SPSS One-Sample t-test of pass9th Variable: Basic Statistics

	N	Mean	Std. Deviation	Std. Error Mean
Passed 9th Grade	94	65.86	13.609	1.404

The null hypothesis is:  $H_0$ : The true mean of pass9th variable = 70; the SPSS program actually calculates the test statistics for  $H_a$ : mean of pass9th is not > 70, so the t-test will be negative because  $M - \mu$  (65.86 – 70) is negative; so we used its absolute

value of  $t = 2.948$  to compare against the  $t_{CV}$  obtained for Table A2 in appendix A. Table 6.2.2 shows the SPSS analysis results of the one-sample  $t$  test with a test value of 70.

Table 6.2.2 SPSS One-Sample  $t$ -test of *pass9th* Variable:  $t$ -test Analysis

	Test Value = 70					
	$t$	$df$	Sig. (2-tailed)	Mean Difference	95% Confidence Interval of the Difference	
					Lower	Upper
Passed 9th Grade	-2.948	93	.004	-4.138	-6.93	-1.35

The  $t$  critical value obtained from the *one-tailed*  $t$ -distribution table in the is 1.661 (Table A2:  $df = 90+$ ,  $\alpha = 0.05$ ) and since the  $t$  statistics obtained from Table 6.2.2 is 2.948 (absolute value used, so -2.948 becomes 2.948) is greater than  $t_{CV} = 1.661$ ; we **reject the null hypothesis** in favor of the alternate hypothesis that there is a significant difference between the test value of 70 and the sample mean for the *pass9th* variable. The critical value used to compute the CI95 is  $t_{CV} = 1.9858$  (Table A3:  $df = 93$ ,  $\alpha = 0.05$ , two-tailed). We could also say that *the mean for the pass9th variable of 65.86 is significantly less than 70*.

**We reject the null hypothesis** for the following reasons (any one):

1. The significance  $p$ -value =  $(0.004 \times 2) = 0.008 < 0.05$
2. The absolute value of the  $t$ -test of 2.948  $>$   $t$ -value of 1.661 (appendix)
3. The confidence interval (-6.93 to -1.35) of the mean difference does *not* contain zero. The  $CI_{\text{Difference}} = -4.138 \pm 1.9858(1.404)$ .